Chapter 7

Appell-Lerch sums

7.1 The Jacobi theta function

Definition 7.1.1. We define the *Jacobi theta function* $\vartheta : \mathbb{C} \times \mathbb{H} \longrightarrow \mathbb{C}$ by

$$\vartheta(z;\tau) := \sum_{\nu \in \frac{1}{2}+\mathbb{Z}} (-1)^{\nu-1/2} e^{\pi i \nu^2 \tau + 2\pi i \nu z}.$$

We often omit the variable τ when it is not varying and simply write $\vartheta(z)$.

Proposition 7.1.2. *Up to a multiplicative constant, ϑ (as a function of z) is the unique entire function satisfying the elliptic transformation properties*

$$\vartheta(z+1) = -\vartheta(z) \qquad \textit{and} \qquad \vartheta(z+\tau) = -e^{-\pi i \tau - 2\pi i z}\vartheta(z).$$

Further ϑ is odd and the only zeros of ϑ are simple zeros in $\mathbb{Z}\tau + \mathbb{Z}$.

Proof. Using that $e^{2\pi i \nu} = -1$ holds for all $\nu \in \frac{1}{2} + \mathbb{Z}$ we immediately get

$$\vartheta(z+1) = \sum_{\nu \in \frac{1}{2}+\mathbb{Z}} (-1)^{\nu-1/2} e^{\pi i \nu^2 \tau + 2\pi i \nu (z+1)} = - \sum_{\nu \in \frac{1}{2}+\mathbb{Z}} (-1)^{\nu-1/2} e^{\pi i \nu^2 \tau + 2\pi i \nu z} = -\vartheta(z).$$

Further replacing ν by $\nu+1$ we obtain

$$\begin{aligned} \vartheta(z) &= \sum_{\nu \in \frac{1}{2}+\mathbb{Z}} (-1)^{\nu+1/2} e^{\pi i (\nu+1)^2 \tau + 2\pi i (\nu+1) z} \\ &= -e^{\pi i \tau + 2\pi i z} \sum_{\nu \in \frac{1}{2}+\mathbb{Z}} (-1)^{\nu-1/2} e^{\pi i \nu^2 \tau + 2\pi i \nu (z+\tau)} = -e^{\pi i \tau + 2\pi i z}\vartheta(z+\tau) \end{aligned}$$

and so $\vartheta(z+\tau) = -e^{-\pi i \tau - 2\pi i z}\vartheta(z)$. Conversely, if an entire function f satisfies

$$f(z+1) = -f(z) \qquad \text{and} \qquad f(z+\tau) = -e^{-\pi i \tau - 2\pi i z} f(z),$$

then $e^{-\pi iz}f(z)$ is 1-periodic and so we can write it as

$$e^{-\pi iz}f(z)=\sum_{n\in\mathbb{Z}}a_n\,e^{2\pi inz}.$$

We may set $a_n=e^{\pi i(n+\frac{1}{2})^2\tau}b_{n+\frac{1}{2}}$ and then we have

$$f(z)=\sum_{n\in\mathbb{Z}}b_{n+\frac{1}{2}}\,e^{\pi i(n+\frac{1}{2})^2\tau+2\pi i(n+\frac{1}{2})z}=\sum_{\nu\in\frac{1}{2}+\mathbb{Z}}b_\nu\,e^{\pi i\nu^2\tau+2\pi i\nu z}.$$

Using this formula we get

$$e^{\pi i\tau+2\pi iz}f(z+\tau)=\sum_{\nu\in\frac{1}{2}+\mathbb{Z}}b_\nu\,e^{\pi i(\nu+1)^2\tau+2\pi i(\nu+1)z}=\sum_{\nu\in\frac{1}{2}+\mathbb{Z}}b_{\nu-1}\,e^{\pi i\nu^2\tau+2\pi i\nu z}$$

and $f(z)=-e^{\pi i\tau+2\pi iz}f(z+\tau)$ then gives $b_\nu=-b_{\nu-1}$ and $b_\nu=(-1)^{\nu-1/2}b_{1/2}$. Hence we have obtained

$$f(z)=b_{1/2}\sum_{\nu\in\frac{1}{2}+\mathbb{Z}}(-1)^{\nu-1/2}e^{\pi i\nu^2\tau+2\pi i\nu z}=b_{1/2}\,\vartheta(z),$$

as desired. Replacing ν by $-\nu$ in the definition of ϑ immediately gives $\vartheta(-z)=-\vartheta(z)$. Hence ϑ has a zero for $z=0$ and by the elliptic transformation properties also in all points of $\mathbb{Z}\tau+\mathbb{Z}$. What remains to be shown is that these zeros are simple and that there are no further zeros. For this we count the number of zeros of ϑ inside the (fundamental) parallelogram $P_p:=p+(0,1)\tau+(0,1)$, where $p\in\mathbb{C}$ is such that there are no zeros on the boundary of P_p. From complex analysis we know that the number of zeros (counting multiplicities) of a holomorphic function can be computed by integrating the logarithmic derivative along the boundary, so the number of zeros of ϑ inside P_p equals

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial P_p}\frac{\vartheta'(z)}{\vartheta(z)}\,dz&=\frac{1}{2\pi i}\left(\int_p^{p+1}+\int_{p+1}^{p+\tau+1}-\int_{p+\tau}^{p+\tau+1}-\int_p^{p+\tau}\right)\frac{\vartheta'(z)}{\vartheta(z)}\,dz\\&=\frac{1}{2\pi i}\int_p^{p+1}\left(\frac{\vartheta'(z)}{\vartheta(z)}-\frac{\vartheta'(z+\tau)}{\vartheta(z+\tau)}\right)dz-\frac{1}{2\pi i}\int_p^{p+\tau}\left(\frac{\vartheta'(z)}{\vartheta(z)}-\frac{\vartheta'(z+1)}{\vartheta(z+1)}\right)dz.\end{aligned}$$

Differentiating the transformation properties of ϑ we find that ϑ'/ϑ is 1-periodic and that

$$\vartheta'(z+\tau)=2\pi i\,e^{-\pi i\tau-2\pi iz}\vartheta(z)-e^{-\pi i\tau-2\pi iz}\vartheta'(z),\quad\text{so}\qquad\frac{\vartheta'(z)}{\vartheta(z)}-\frac{\vartheta'(z+\tau)}{\vartheta(z+\tau)}=2\pi i.$$

Hence the number of zeros of ϑ inside P_p is $\int_p^{p+1}dz=1$. Since P_p contains exactly one point from $\mathbb{Z}\tau+\mathbb{Z}$, this means that we have already found all zeros and that these zeros are all simple. □

Proposition 7.1.3. *The function ϑ satisfies the modular transformation properties*

$$\vartheta(z;\tau+1)=\zeta_8\,\vartheta(z;\tau)\qquad \textit{and} \qquad \vartheta\Big(\frac{z}{\tau};-\frac{1}{\tau}\Big)=-i\sqrt{-i\tau}\,e^{\pi i z^2/\tau}\vartheta(z;\tau),$$

where (as usual) $\zeta_N:=e^{2\pi i/N}$.

Proof. For all $\nu\in\frac12+\mathbb{Z}$ we have that $\nu^2\in\frac14+2\mathbb{Z}$ and so $e^{\pi i\nu^2}=\zeta_8$. From this the first transformation property follows immediately. Further, using

$$\int_{\mathbb{R}} e^{-\pi i u^2/\tau-2\pi i u v}du=\sqrt{-i\tau}\,e^{\pi i v^2\tau} \tag{7.1}$$

(Proposition 6.4.11 with $A=1\in M_{1,1}(\mathbb{R})$ and $P=1$) we get that for the Fourier transform of

$$f(v)=e^{-\pi i v}e^{-\pi i(v+\frac12)^2/\tau+2\pi i(v+\frac12)x}\qquad (x\in\mathbb{R})$$

we have

$$\begin{aligned}\big(\mathcal{F}f\big)(v)&=\int_{\mathbb{R}}e^{-\pi i u-\pi i(u+\frac12)^2/\tau+2\pi i(u+\frac12)x-2\pi i u v}du\\ &\overset{u\to u-\frac12}{=}e^{\pi i(v+\frac12)}\int_{\mathbb{R}}e^{-\pi i u^2/\tau-2\pi i u(-x+v+\frac12)}du\\ &=\sqrt{-i\tau}\,e^{\pi i(v+\frac12)}e^{\pi i(-x+v+\frac12)^2\tau}=i\sqrt{-i\tau}\,e^{\pi i x^2\tau}e^{\pi i v}e^{\pi i(v+\frac12)^2\tau-2\pi i(v+\frac12)x\tau}.\end{aligned}$$

Using Poisson summation we then find

$$\vartheta(x;-1/\tau)=\sum_{n\in\mathbb{Z}}f(n)=\sum_{n\in\mathbb{Z}}\big(\mathcal{F}f\big)(n)=i\sqrt{-i\tau}\,e^{\pi i x^2\tau}\vartheta(-x\tau;\tau).$$

This identity holds for all $x\in\mathbb{R}$ and by analytic continuation for all $x\in\mathbb{C}$. Now setting $x=z/\tau$ and using that ϑ is odd we get the desired result. □

Proposition 7.1.4. *For all $\tau\in\mathbb{H}$ we have*

$$\frac{1}{2\pi i}\vartheta'(0;\tau)=\eta(\tau)^3.$$

Remark 7.1.5. Here $'$ means the derivative with respect to z. Further η is the Dedekind η-function (as in remark 3.5.9), which satisfies the modular transformation properties

$$\eta(\tau+1)=\zeta_{24}\,\eta(\tau)\qquad\text{and}\qquad \eta(-1/\tau)=\sqrt{-i\tau}\,\eta(\tau).$$

Proof of Proposition 7.1.4. Since η has no zeros in $\mathbb{H}$, the function $f : \mathbb{H} \longrightarrow \mathbb{C}$ given by $f(\tau) = \vartheta'(0;\tau)/\big(2\pi i\eta(\tau)^3\big)$ is holomorphic on $\mathbb{H}$. Using

$$\frac{1}{2\pi i}\vartheta'(0;\tau) = \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \nu\,(-1)^{\nu-\frac{1}{2}}\,e^{\pi i\nu^2\tau} = q^{1/8}\big(1+\mathcal{O}(q)\big),$$
$$\eta(\tau)^3 = q^{1/8}\big(1+\mathcal{O}(q)\big)$$

we get $f(\tau) = 1+\mathcal{O}(q)$ and so f is also holomorphic at ∞. Further, taking the derivative (with respect to z) and then setting $z = 0$ in the modular transformation properties of ϑ immediately gives

$$\vartheta'(0;\tau+1) = \zeta_8\,\vartheta'(0;\tau) \qquad \text{and} \qquad \vartheta'(0;-1/\tau) = (-i\tau)^{3/2}\,\vartheta'(0;\tau).$$

Combining these modular transformation properties with those of η, we see that f transforms as a modular forms of weight 0 and so we have $f \in M_0(\Gamma_1) = \mathbb{C}$. Hence we find $f \equiv 1$, which gives the desired result. $\square$

Proposition 7.1.6 (Jacobi triple product identity)**.** *We have*

$$\vartheta(z;\tau) = -q^{1/8}\zeta^{-1/2}\prod_{n=1}^{\infty}(1-q^n)(1-\zeta q^{n-1})(1-\zeta^{-1}q^n) \qquad (q = e^{2\pi i\tau},\ \zeta = e^{2\pi iz}).$$

Proof. We define ϑ^* as the right hand side:

$$\vartheta^*(z) = \vartheta^*(z;\tau) := -q^{1/8}\zeta^{-1/2}\prod_{n=1}^{\infty}(1-q^n)(1-\zeta q^{n-1})(1-\zeta^{-1}q^n).$$

Replacing z by $z+1$ immediately gives $\vartheta^*(z+1) = -\vartheta^*(z)$ and if we replace z by $z+\tau$, we have to replace ζ by ζq. Hence we find

$$\vartheta^*(z+\tau) = -q^{-3/8}\zeta^{-1/2}\prod_{n=1}^{\infty}(1-q^n)(1-\zeta q^n)(1-\zeta^{-1}q^{n-1}),$$

and using

$$\prod_{n=1}^{\infty}(1-\zeta q^n) = \frac{1}{1-\zeta}\prod_{n=1}^{\infty}(1-\zeta q^{n-1}),$$
$$\prod_{n=1}^{\infty}(1-\zeta^{-1}q^{n-1}) = (1-\zeta^{-1})\prod_{n=1}^{\infty}(1-\zeta^{-1}q^n)$$

we then get

$$\vartheta^*(z+\tau) = q^{-1/2}\frac{1-\zeta^{-1}}{1-\zeta}\,\vartheta^*(z) = -q^{-1/2}\zeta^{-1}\,\vartheta^*(z) = -e^{-\pi i\tau-2\pi iz}\,\vartheta^*(z).$$

Using Proposition 7.1.2 we find that $\vartheta^*(z;\tau) = c(\tau)\,\vartheta(z;\tau)$, where c doesn't depend on z. What remains to be shown is that $c \equiv 1$, for which it suffices to prove that

$$\lim_{z\to 0} \frac{\vartheta(z;\tau)}{z} = \lim_{z\to 0} \frac{\vartheta^*(z;\tau)}{z} \neq 0.$$

The left hand side is $\vartheta'(0;\tau) = 2\pi i\,\eta(\tau)^3$ and the expression on the right equals

$$\begin{aligned} -q^{1/8} \lim_{z\to 0} &\left[\frac{(1-e^{2\pi i z})}{z} \zeta^{-1/2} \prod_{n=1}^{\infty} (1-q^n)(1-\zeta q^n)(1-\zeta^{-1}q^n) \right] \\ &= -q^{1/8} \lim_{z\to 0} \frac{(1-e^{2\pi i z})}{z} \prod_{n=1}^{\infty} (1-q^n)^3 = 2\pi i\,\eta(\tau)^3. \end{aligned}$$

Hence they are equal and nonzero, as desired. □

Corollary 7.1.7. *We have*

$$\eta(\tau) = \sum_{n=1}^{\infty} \chi(n)\, q^{\frac{1}{24}n^2},$$

where χ is the Dirichlet character modulo 12 with $\chi(\pm 1) = 1$ and $\chi(\pm 5) = -1$.

Remark 7.1.8. In terms of the Kronecker symbol (see Definition 5.5.11) we have $\chi(n) = \left(\frac{12}{n}\right)$.

Proof of Corollary 7.1.7. If we replace τ by 3τ and set $z = \tau$ in the Jacobi triple product identity we find

$$\vartheta(\tau;3\tau) = -q^{-1/8} \prod_{n=1}^{\infty} (1-q^{3n})(1-q^{3n-2})(1-q^{3n-1}) = -q^{-1/8} \prod_{n=1}^{\infty} (1-q^n),$$

which equals $-q^{-1/6}\eta(\tau)$. Hence we have

$$\eta(\tau) = -q^{1/6}\,\vartheta(\tau;3\tau) = -\sum_{\nu\in\frac{1}{2}+\mathbb{Z}} (-1)^{\nu-\frac{1}{2}} q^{\frac{3}{2}\nu^2+\nu+\frac{1}{6}} = \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} (-1)^{\nu+\frac{1}{2}} q^{\frac{1}{24}(6\nu+2)^2}$$

and substituting $\nu = n - \frac{1}{2}$ gives

$$\begin{aligned} \eta(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{24}(6n-1)^2} &= \sum_{n=1}^{\infty} (-1)^n q^{\frac{1}{24}(6n-1)^2} + \sum_{n=-\infty}^{0} (-1)^n q^{\frac{1}{24}(6n-1)^2} \\ &= \sum_{\substack{m=1 \\ m\equiv 5 \pmod 6}}^{\infty} \chi(m)\, q^{\frac{1}{24}m^2} + \sum_{\substack{m=1 \\ m\equiv 1 \pmod 6}}^{\infty} \chi(m)\, q^{\frac{1}{24}m^2}, \end{aligned}$$

where in the last step we have substituted $m = 6n-1$ in the first sum and $m = -6n+1$ in the second. This gives the desired result. □